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1984 J. Phys. A: Math. Gen. 17 939

(http://iopscience.iop.org/0305-4470/17/5/018)

J. Phys. A: Math. Gen. 17 (1984) 939–955. Printed in Great Britain

Division algebras, (pseudo)orthogonal groups and spinors

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Received 1 September 1983

Abstract. The groups SO($\nu-1$), SO(ν), SO($\nu+1$), SO($\nu+1,1$) and SO($\nu+2,2$) ($\nu=1,2,4,8$) and their spin representations are described in terms of the division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$.

1. Introduction

It has become apparent recently that various aspects of supersymmetric field theories have a natural expression in terms of the division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$ (the real numbers, complex numbers, quaternions and octonions, respectively). Examples involving the complex numbers and quaternions are described by Kugo and Townsend (1982) and in the references they cite; in addition the octonions have been used to describe a spontaneous compactification of supergravity in 11 space–time dimensions (Dereli *et al* 1983) and it seems likely (Duff *et al* 1982) that the quaternions can be used to describe spontaneous compactification in dimensions $d=7$, 8, 9, 10.

Kugo and Townsend (1982) note that for rigid supersymmetry there is a pattern of association between the maximal space–time dimension D of a theory and a division algebra, in which $\mathbb{R}$ is associated with $D=3$, $\mathbb{C}$ with $D=4$ and $\mathbb{H}$ with $D=6$, and they speculate that this pattern could be extended to associate the octonions $\mathbb{O}$ with $D=10$. They relate this pattern to the sequence of isomorphisms which forms the second column of table 1, which they explain in terms of spinors of the various Lorentz groups involved. They also give similar explanations of the other two columns of table 1.

The arguments of Kugo and Townsend really establish homomorphisms, which happen to be isomorphisms because the dimensions match. In this paper we will give a unified treatment of each of the columns of table 1, which is valid for any normed

Table 1. Isomorphisms between pseudo-orthogonal groups Spin(s, t) and groups involving a division algebra $\mathbb{K}$.

	$t=0$	$t=1$	$t=2$
$s-t=1$, $\mathbb{K}=\mathbb{R}$	$\mathrm{Spin}(1)\cong \mathrm{SL}(1,\mathbb{R})$	$\mathrm{Spin}(2,1)\cong \mathrm{SL}(2,\mathbb{R})$	$\mathrm{Spin}(3,2)\cong \mathrm{Sp}(4,\mathbb{R})$
$s-t=2$, $\mathbb{K}=\mathbb{C}$	$\mathrm{Spin}(2)\cong \mathrm{SL}_1(1,\mathbb{C})$	$\mathrm{Spin}(3,1)\cong \mathrm{SL}(2,\mathbb{C})$	$\mathrm{Spin}(4,2)\cong \mathrm{SU}(2,2)$
$s-t=4$, $\mathbb{K}=\mathbb{H}$	$\mathrm{Spin}(4)\cong \mathrm{SL}_1(1,\mathbb{H})\times \mathrm{SL}_1(1,\mathbb{H})$	$\mathrm{Spin}(5,1)\cong \mathrm{SL}(2,\mathbb{H})$	$\mathrm{Spin}(6,2)\cong \mathrm{Sp}(4,\mathbb{H})$

0305-4470/84/050939+17\$02.25

division algebra $\mathbb{K}$. In particular, our arguments can be applied directly to the case of the octonions ($\mathbb{K}=\mathbb{O}$), thus establishing a fourth row of the table (printed as table 2) and verifying the conjecture of Kugo and Townsend.

The ideas used in this unified treatment derive from an unpublished review by Ramond (1976). Ramond deals mainly with finite group elements, which can lead to rather cumbersome expressions when octonions are involved. By considering infinitesimal transformations, i.e. by working with the Lie algebra rather than the Lie group in each case, we can reduce the complications arising from the non-associative nature of octonion multiplication, and so most of this paper will be couched in terms of Lie algebras rather than Lie groups.

Table 2. Groups Spin(s, t) involving octonions.

$s-t=8$ $\mathbb{K}=\mathbb{O}$	$\mathrm{Spin}(8)\cong \mathrm{G}_2\times \mathrm{SL}_1(1,\mathbb{O})\times \mathrm{SL}_1(1,\mathbb{O})$	$\mathrm{Spin}(9,1)\cong \mathrm{SL}(2,\mathbb{O})$	$\mathrm{Spin}(10,2)\cong \mathrm{Sp}(4,\mathbb{O})$

Our notation for Lie groups and Lie algebras is as follows. As usual, SO(s, t) denotes the pseudo-orthogonal group of a space–time with s space and t time dimensions, i.e. the group of $(s+t)\times(s+t)$ real matrices R satisfying $\det R=1$ and $R^{\mathrm{T}}GR=G$ where $G=\mathrm{diag}(-1,\ldots,-1,+1,\ldots,1)$ with s $-$ signs and t $+$ signs. Its Lie algebra, consisting of matrices A satisfying $A^{\mathrm{T}}G+GA=0$, is denoted by $\mathbf{so}(s,t)$ (the initial **s** is unnecessary but not incorrect). If $t=0$ we write simply SO(s) and $\mathbf{so}(s)$. The double cover of $\mathbf{so}(s,t)$ is denoted by Spin(s, t). More generally, if V is a real vector space with a symmetric bilinear form g, SO(V) denotes the group of endomorphisms of V which preserve the form g and $\mathbf{so}(V)$ denotes the Lie algebra of antisymmetric endomorphisms of V (with respect to g).

The pseudo-unitary group SU(s, t) and Lie algebra $\mathbf{su}(s,t)$ are defined similarly: SU(s, t) is the group of $(s+t)\times(s+t)$ complex matrices U satisfying $\det U=1$ and $U^{\dagger}GU=G$, where $U^{\dagger}$ is the hermitian conjugate of U, and $\mathbf{su}(s,t)$ is the Lie algebra of matrices A satisfying $A^{\dagger}G+GA=0$.

The name 'symplectic' and the symbols Sp and **sp** have unfortunately become associated with two distinct families of groups and algebras. We will therefore change the symbol for one of these families and use Sq(n) to denote the group of $n\times n$ quaternionic matrices R satisfying $R^{\dagger}R=I$, where $R^{\dagger}$ denotes the quaternionic hermitian conjugate, and $\mathbf{sq}(n)$ for its Lie algebra, consisting of quaternionic matrices A satisfying $A^{\dagger}=-A$. Then $\mathrm{Sp}(2n,\mathbb{K})$ denotes the symplectic group of $2n\times 2n$ matrices R with entries in $\mathbb{K}$ ($=\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$), satisfying $R^{\dagger}JR=J$ and with $\det R=1$ (if $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$); $\mathbf{sp}(2n,\mathbb{K})$ denotes its Lie algebra, consisting of matrices satisfying $A^{\dagger}J+JA=0$ and $\mathrm{Tr}\,A=0$ (if $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$) ($J=\begin{pmatrix}0 & I\\ -I & 0\end{pmatrix}$).

$\mathrm{SL}(n,\mathbb{K})$ denotes the special linear group of $n\times n$ matrices over $\mathbb{K}$ whose determinant is 1 (if $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$) or has modulus 1 (if $\mathbb{K}=\mathbb{H}$); its Lie algebra $\mathbf{sl}(n,\mathbb{K})$ consists of matrices A whose traces are 0 (if $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$) or have zero real part (if $\mathbb{K}=\mathbb{H}$). If V is any real vector space, $\mathbf{gl}(V)$ denotes the Lie algebra of all endomorphisms of V and $\mathbf{sl}(V)$ denotes the subalgebra of traceless elements.

Sections 4 and 5 contain unified definitions of the antihermitian Lie algebra $\mathbf{sa}(n,\mathbb{K})$ (incorporating $\mathbf{so}(n)$, $\mathbf{su}(n)$ and $\mathbf{sq}(n)$) and of $\mathbf{sp}(n,\mathbb{K})$ and $\mathbf{sl}(n,\mathbb{K})$, which can be extended to the octonionic case $\mathbb{K}=\mathbb{O}$ if $n=2$ or 3.

The direct sum symbol $\oplus$ denotes a direct sum of vector spaces, not necessarily a direct sum of Lie algebras. I denotes an identity matrix.

The dimension of the division algebra $\mathbb{K}$ will always be denoted by ν. Thus ν takes values 1, 2, 4 and 8, corresponding to $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$.

2. Division algebras and orthogonal groups ($s = \nu$ and $\nu - 1$, $t = 0$)

We will be concerned on the one hand with space–time of dimension $d = s + t$, and on the other hand with a composition algebra $\mathbb{K}$. In this section we will consider the case $t = 0$ and show how the orthogonal groups $\mathrm{SO}(\nu)$ and $\mathrm{SO}(\nu-1)$ can be described in terms of the multiplication in the algebra $\mathbb{K}$.

A *composition algebra* (over $\mathbb{R}$) is an algebra $\mathbb{K}$ which has a non-degenerate quadratic form, which we will denote by $x \mapsto |x|^2$, satisfying

$$|xy|^2 = |x|^2|y|^2, \qquad x, y \in \mathbb{K}. \tag{2.1}$$

Such an algebra can always be assumed to have an identity element (Curtis 1963, Jacobson 1958) and therefore to have $\mathbb{R}$ embedded in it. The quadratic form $|x|^2$ induces an inner product in $\mathbb{K}$; the subspace orthogonal to $\mathbb{R}$ will be denoted by $\mathbb{K}'$. The conjugation which fixes every element of $\mathbb{R}$ and multiplies every element of $\mathbb{K}'$ by -1 is denoted by $x \mapsto \bar{x}$; it satisfies

$$\overline{xy} = \bar{y}\bar{x} \tag{2.2}$$

and

$$x\bar{x} = |x|^2. \tag{2.3}$$

We write

$$\operatorname{Re} x = \tfrac{1}{2}(x + \bar{x}); \tag{2.4}$$

then the inner product is given by

$$\langle x, y\rangle = \operatorname{Re}(x\bar{y}) = \operatorname{Re}(\bar{x}y). \tag{2.5}$$

It satisfies

$$\langle x, yz\rangle = \langle x\bar{z}, y\rangle. \tag{2.6}$$

Any composition algebra $\mathbb{K}$ is *alternative*, i.e. the associator

$$[x, y, z] = x(yz) - (xy)z \tag{2.7}$$

is an alternating function of $x, y, z \in \mathbb{K}$ (Curtis 1963, Jacobson 1958). If the quadratic form $|x|^2$ is positive definite, it follows immediately from (2.1) that $\mathbb{K}$ is a division algebra, i.e.

$$xy = 0 \quad \Rightarrow \quad x = 0 \quad \text{or} \quad y = 0. \tag{2.8}$$

By 'division algebra' we shall always mean such a positive-definite composition algebra (or *normed* division algebra). Hurwitz's theorem (Curtis 1963) states that the only such algebras are $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ and $\mathbb{O}$. We shall sometimes denote these by $\mathbb{K}_\nu$ ($\nu = 1, 2, 4, 8$), labelling the algebra by its dimension over $\mathbb{R}$.

From (2.1) it follows that if $u \in \mathbb{K}$ has norm 1, both left multiplication and right multiplication by u are orthogonal maps of $\mathbb{K}$. We denote these by L_u and R_u. As the first column of table 1 indicates, these maps generate the whole rotation group

SO($\mathbb{K}$) of $\mathbb{K}$, but the details depend on the particular case. They are most simply described in terms of the Lie algebra of the group, i.e. the Lie algebra $\mathbf{so}(\mathbb{K})$ of antisymmetric maps $T: \mathbb{K} \to \mathbb{K}$.

A subgroup of the rotation group of $\mathbb{K}$ is the automorphism group Aut $\mathbb{K}$; its Lie algebra is the algebra Der $\mathbb{K}$ of derivations of $\mathbb{K}$, i.e. the set of linear maps $D: \mathbb{K} \to \mathbb{K}$ satisfying

$$D(xy) = (Dx)y + x(Dy). \tag{2.9}$$

These Lie algebras are

$$\text{Der}\,\mathbb{R} = \text{Der}\,\mathbb{C} = 0, \qquad \text{Der}\,\mathbb{H} = C(\mathbb{H}') \cong \mathbf{sq}(1), \qquad \text{Der}\,\mathbb{O} = G_2 \tag{2.10}$$

where $C(\mathbb{H}')$ is defined below.

It follows from the alternative law that for any $a, b \in \mathbb{K}$ the linear map $D(a, b)$ defined by

$$D(a, b)x = [a, b, x] + \tfrac{1}{3}[[a, b], x] \tag{2.11}$$

is a derivation of $\mathbb{K}$. It is generated by the left and right multiplication maps L_d and R_d with $d \in \mathbb{K}'$; in fact it is generated by the commutator maps $C_d = L_d - R_d$, for

$$D(a, b) = \tfrac{1}{6}([C_a, C_b] + C_{[a,b]}). \tag{2.12}$$

Such a derivation, and any sum of such derivations, is called an *inner derivation*. It can be shown (Schafer 1966) that every derivation of a composition algebra is inner.

In general the derivations and the multiplication maps L_x and R_x give all antisymmetric maps of $\mathbb{K}$:

$$\mathbf{so}(\mathbb{K}) = \text{Der}\,\mathbb{K} + L(\mathbb{K}') + R(\mathbb{K}') \tag{2.13}$$

where $L(\mathbb{K}')$ is the set of all L_a with $a \in \mathbb{K}'$ and $R(\mathbb{K}')$ is the set of all R_a. Antisymmetric maps of $\mathbb{K}'$ are obtained by restricting the multiplication maps to the commutator maps C_a:

$$\mathbf{so}(\mathbb{K}') = \text{Der}\,\mathbb{K} + C(\mathbb{K}'). \tag{2.14}$$

The sums in (2.13) and (2.14) are not necessarily direct sums, but in all cases the Lie brackets are given by

$$\begin{aligned} &[D, L_a] = L_{Da}, \qquad [D, R_a] = R_{Da}, \\ &[L_a, L_b] = 2D(a, b) + \tfrac{1}{3}L_{[a,b]} + \tfrac{2}{3}R_{[a,b]}, \\ &[R_a, R_b] = 2D(a, b) - \tfrac{2}{3}L_{[a,b]} - \tfrac{1}{3}R_{[a,b]}, \\ &[L_a, R_b] = -D(a, b) + \tfrac{1}{3}L_{[a,b]} - \tfrac{1}{3}R_{[a,b]} \end{aligned} \tag{2.15}$$

($D \in \text{Der}\,\mathbb{K}$; $a, b \in \mathbb{K}'$).

If $\mathbb{K}$ is neither commutative nor associative the sums in (2.13) and (2.14) are direct sums (as vector spaces, not as Lie algebras). Thus $\mathbb{K} = \mathbb{O}$ gives

$$\mathbf{so}(8) \cong G_2 \oplus \mathbb{O}' \oplus \mathbb{O}', \tag{2.16}$$

$$\mathbf{so}(7) \cong G_2 \oplus \mathbb{O}'. \tag{2.17}$$

If $\mathbb{K}$ is associative, it follows from (2.11) and the fact that all derivations are inner that $\text{Der}\,\mathbb{K} \subset L(\mathbb{K}') + R(\mathbb{K}')$; also $L(\mathbb{K}')$ and $R(\mathbb{K}')$ are isomorphic and commuting subalgebras. Thus $\mathbb{K}=\mathbb{H}$ gives

$$\mathbf{so}(4)\cong\mathbf{sq}(1)\oplus\mathbf{sq}(1), \tag{2.18}$$

$$\mathbf{so}(3)\cong\mathbf{sq}(1), \tag{2.19}$$

and the sum in (2.18) is a direct sum of Lie algebras.

If in addition $\mathbb{K}$ is commutative, then $L(\mathbb{K}')=R(\mathbb{K}')$ and $C(\mathbb{K}')=0$; thus $\mathbb{K}=\mathbb{C}$ gives

$$\mathbf{so}(2)=\mathbf{u}(1), \tag{2.20}$$

$$\mathbf{so}(1)=0. \tag{2.21}$$

3. 2×2 matrices: orthogonal and Lorentz groups ($s=\nu+1$, $t=0$ and 1)

Let $H_n(\mathbb{K})$ denote the set of hermitian $n\times n$ matrices with entries in the division algebra $\mathbb{K}$, i.e. the set of matrices X satisfying $X^\dagger=X$ where $X^\dagger=\bar{X}^{\mathrm{T}}$ and the bar denotes conjugation in $\mathbb{K}$. Let $A_n(\mathbb{K})$ denote the set of antihermitian matrices, defined by the condition $X^\dagger=-X$, and let $L_n(\mathbb{K})=A_n(\mathbb{K})\oplus H_n(\mathbb{K})$ be the set of all $n\times n$ matrices. If $\mathbb{K}$ is associative $A_n(\mathbb{K})$ and $L_n(\mathbb{K})$ are Lie algebras with the Lie bracket given by the commutator. They each have a centre consisting of multiples of the identity matrix; the quotient of the Lie algebra by this centre will be denoted by $\mathbf{sa}(n,\mathbb{K})$ and $\mathbf{sl}(n,\mathbb{K})$ respectively. These are contained in $A_n(\mathbb{K})$ and $L_n(\mathbb{K})$ as subalgebras defined by conditions on the trace of the matrices: for $\mathbb{K}=\mathbb{R}$ and $\mathbb{H}$ the centre of $A_n(\mathbb{K})$ is zero and so $\mathbf{sa}(n,\mathbb{R})=\mathbf{so}(n)$ and $\mathbf{sa}(n,\mathbb{H})=\mathbf{sq}(n)$ coincide with $A_n(\mathbb{R})$ and $A_n(\mathbb{H})$, while $\mathbf{sa}(n,\mathbb{C})=\mathbf{su}(n)$ consists of matrices with zero trace; and the special linear algebras $\mathbf{sl}(n,\mathbb{K})$ are defined by the condition $\operatorname{Tr}X=0$ if $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$ and by the condition $\operatorname{Re}\operatorname{Tr}X=0$ if $\mathbb{K}=\mathbb{H}$.

In this section we will obtain an alternative definition of $\mathbf{sa}(n,\mathbb{K})$ and $\mathbf{sl}(n,\mathbb{K})$ which can be extended to the case of a non-associative $\mathbb{K}$ if $n=2$ or 3.

For $n=2$ or 3, and for all n if $\mathbb{K}$ is associative, $H_n(\mathbb{K})$ forms a Jordan algebra with the product given by the anticommutator; i.e. if we define

$$X\cdot Y=\tfrac{1}{2}(XY+YX) \tag{3.1}$$

then $X\cdot Y$ is a commutative but non-associative product satisfying

$$X\cdot(X^2\cdot Y)=X^2\cdot(X\cdot Y). \tag{3.2}$$

We will examine the Lie algebra of derivations of this Jordan algebra. If $\mathbb{K}$ is associative, the derivations are all of the form

$$X\mapsto\operatorname{ad}A(X)=[A,X] \tag{3.3}$$

for some antihermitian matrix A. This is the zero derivation if and only if $A=\lambda 1$ where λ belongs to the centre of $\mathbb{K}$; thus the derivation algebra of $H_n(\mathbb{K})$ is

$$\operatorname{Der}H_n(\mathbb{K})\cong\mathbf{sa}(n,\mathbb{K}) \tag{3.4}$$

if $\mathbb{K}$ is associative.

The matrix identity which makes ad A a derivation of $H_n(\mathbb{K})$, namely

$$[A,\{X,Y\}]=\{[A,X],Y\}+\{X,[A,Y]\}, \tag{3.5}$$

which is valid for all $n\times n$ matrices if $\mathbb{K}$ is associative, holds for restricted classes of

2×2 and 3×3 matrices if $\mathbb{K}$ is only alternative: in particular, it holds if X and Y are hermitian and A is antihermitian (for 2×2 matrices) or antihermitian and traceless (for 3×3 matrices). Thus for any division algebra $\mathbb{K}$, ad A is a derivation of $H_2(\mathbb{K})$ if $A\in A_2(\mathbb{K})$. Derivations of $\mathbb{K}$ also act as derivations of $H_2(\mathbb{K})$ by acting on the entries in the matrices. These are all the derivations of $H_2(\mathbb{K})$, so we have

$$\text{Der}\, H_2(\mathbb{K}) = \text{ad}\, A_2(\mathbb{K}) + \text{Der}\, \mathbb{K}. \tag{3.6}$$

We can decompose the space of antihermitian 2×2 matrices as

$$A_2(\mathbb{K}) = A_2'(\mathbb{K}) \oplus \mathbb{K}'I$$

where $A_2'(\mathbb{K})$ is the subspace of traceless matrices and $\mathbb{K}'I$ is the subspace of multiples of the identity matrix. But for $a\in\mathbb{K}'$, $\text{ad}(aI)$ acts on an element of $H_2(\mathbb{K})$ by acting as C_a on each entry in the matrix; thus $\text{ad}(\mathbb{K}'I) = C(\mathbb{K}')$. Also ad $A_2'(\mathbb{K}) = A_2'(\mathbb{K})$ since $H_2(\mathbb{K})$ is an irreducible set. Using (2.14) we now have

$$\text{Der}\, H_2(\mathbb{K}) = A_2'(\mathbb{K}) \oplus \mathbf{so}(\mathbb{K}'). \tag{3.7}$$

We will use (3.7), in the light of (3.4), to define the Lie algebra $\mathbf{sa}(2,\mathbb{K})$ for any division algebra $\mathbb{K}$. If $\mathbb{K}$ is associative the Lie bracket is given by the matrix commutator; this is a consequence of the Jacobi identity for matrices. Like (3.5), this can be used also for matrices with entries in a composition algebra, but now an extra term must be added:

$$[A,[B,X]]-[B,[A,X]] = [[A,B],X] + E(A,B)X \tag{3.8}$$

where $E(A,B)\in\mathbf{so}(\mathbb{K}')$ is given by

$$E(A,B)x = \sum_{ij} [a_{ij}, b_{ji}, x] \qquad (x\in\mathbb{K}). \tag{3.9}$$

This holds if A and B are antihermitian and traceless and X is hermitian. Thus we have

$$\mathbf{sa}(2,\mathbb{K}) = A_2'(\mathbb{K}) \oplus \mathbf{so}(\mathbb{K}') \tag{3.10}$$

in which $\mathbf{so}(\mathbb{K}')$ is a subalgebra; the Lie bracket of $T\in\mathbf{so}(\mathbb{K}')$ and a matrix $A\in A_2'(\mathbb{K})$ is given by the action of T on the entries in A; and the Lie bracket between two matrices in $A_2'(\mathbb{K})$ is

$$[A,B] = (AB-BA-aI) \oplus (C_a + E(A,B)) \tag{3.11}$$

where $a = \frac{1}{2}\text{Tr}(AB-BA)$.

We can give another characterisation of Der $H_2(\mathbb{K})$ by describing the Jordan algebra structure of $H_2(\mathbb{K})$ more explicitly. Let

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad S(x) = \begin{pmatrix} 0 & x \\ \bar{x} & 0 \end{pmatrix} \qquad (x\in\mathbb{K}) \tag{3.12}$$

and let V be the subspace of $H_2(\mathbb{K})$ spanned by P and the $S(x)$, with inner product

$$g(\alpha P + S(x), \beta P + S(y)) = \alpha\beta + \langle x, y\rangle \qquad (\alpha,\beta\in\mathbb{R};\ x,y\in\mathbb{K}). \tag{3.13}$$

Then $H_2(\mathbb{K}) = \mathbb{R}I \oplus V$, I acts as an identity in the Jordan algebra, and the Jordan product of two elements of V is

$$v\cdot w = g(v,w)I. \tag{3.14}$$

Thus as a Jordan algebra $H_2(\mathbb{K})$ is isomorphic to the Jordan algebra $J(V)$ defined on the vector space $\mathbb{R}\oplus V$ by the anticommutator in the Clifford algebra of V. It is easy to see that the set of derivations of this Jordan algebra is just the set of antisymmetric linear maps of V:

$$\mathrm{Der}\, H_2(\mathbb{K}) \cong \mathbf{so}(V). \tag{3.15}$$

From (3.4) and (3.15) we have

$$\mathbf{sa}(2, \mathbb{K}_\nu) \cong \mathbf{so}(\nu+1) \tag{3.16}$$

which incorporates the isomorphisms

$$\mathbf{su}(2) \cong \mathbf{so}(3), \qquad \mathbf{sq}(2) \cong \mathbf{so}(5) \tag{3.17}$$

and the identification of $\mathbf{sa}(2, \mathbb{O})$ as $\mathbf{so}(9)$.

The *structure algebra* (or *Lie multiplication algebra*) of a Jordan algebra $\mathbb{J}$ is the Lie subalgebra of $\mathbf{gl}(\mathbb{J})$ generated by the Jordan multiplication maps L_X, i.e. $L_X(Y) = X\cdot Y$. It follows from the Jordan identity (3.2) that $[L_X, L_Y]$ is a derivation of $\mathbb{J}$. Conversely, if $\mathbb{J}$ is semisimple every derivation is a sum of derivations of this form (Schafer 1966 p 22). If $\mathbb{J}$ has an identity no non-zero L_X is a derivation; hence the structure algebra is

$$\mathrm{Str}\, \mathbb{J} = \mathrm{Der}\, \mathbb{J} \oplus L(\mathbb{J}). \tag{3.18}$$

The multiples of I in $L(\mathbb{J})$ will belong to the centre of Str $\mathbb{J}$; we will factor out this subspace to obtain a reduced structure algebra Str′ $\mathbb{J}$. In the case $\mathbb{J} = H_2(\mathbb{K})$ we can use (3.7) to obtain the vector space structure

$$\mathrm{Str}'\, H_2(\mathbb{K}) \cong H_2'(\mathbb{K}) \oplus A_2'(\mathbb{K}) \oplus \mathbf{so}(\mathbb{K}') \cong L_2'(\mathbb{K}) \oplus \mathbf{so}(\mathbb{K}') \tag{3.19}$$

where $H_2'(\mathbb{K})$ and $L_2'(\mathbb{K})$ denote the traceless subspaces of $H_2(\mathbb{K})$ and $L_2(\mathbb{K})$. If $\mathbb{K}$ is associative the subalgebra $\mathbf{so}(\mathbb{K}')$ can be identified with a set of multiples of the identity matrix; the matrix identities (3.5) and

$$\{X, \{Y, Z\}\} - \{Y\{X, Z\}\} = [[X, Y], Z], \tag{3.20}$$

together with the Jacobi identity, then show that the Lie bracket in Str′ $H_2(\mathbb{K})$ is given by the matrix commutator. Thus we have a Lie algebra isomorphism

$$\mathrm{Str}'\, H_2(\mathbb{K}) \cong \mathbf{sl}(2, \mathbb{K}) \tag{3.21}$$

if $\mathbb{K}$ is associative.

We now proceed as we did for $\mathbf{sa}(2, \mathbb{K})$ and use (3.21) to define $\mathbf{sl}(2, \mathbb{K})$ for any composition algebra $\mathbb{K}$. Like the Jacobi identity, (3.20) can be modified so as to hold for matrices over any composition algebra $\mathbb{K}$:

$$\{X, \{Y, Z\}\} - \{Y, \{X, Z\}\} = [[X, Y], Z] + E(X, Y)Z \tag{3.22}$$

where $E(X, Y)$ is defined as in (3.9). This holds if X, Y and Z are hermitian. Thus we have

$$\mathbf{sl}(2, \mathbb{K}) = L_2'(\mathbb{K}) \oplus \mathbf{so}(\mathbb{K}'), \tag{3.23}$$

the Lie brackets being the same as in $\mathbf{sa}(2, \mathbb{K})$—in particular, the Lie bracket between any two matrices in $L_2'(\mathbb{K})$ is still given by (3.11).

The action on $H_2(\mathbb{K})$ of $L_2'(\mathbb{K})$, regarded as a subspace of the structure algebra Str′ $H_2(\mathbb{R})$, can be summarised as

$$X \mapsto MX + XM^\dagger \qquad (X \in H_2(\mathbb{K}), M \in L_2'(\mathbb{K})). \tag{3.24}$$

If $\mathbb{K}$ is commutative, this map is an infinitesimal generator of the group of transformations $X \mapsto UXU^\dagger$, with $\det U = 1$ corresponding to $\operatorname{Tr} M = 0$. These preserve $\det X$, which is a Lorentzian quadratic form on $H_2(\mathbb{K})$. For a general $\mathbb{K}$, it follows from the composition (3.19) of $\operatorname{Str}' H_2(\mathbb{K})$, together with the form (3.14) for the Jordan product in $H_2(\mathbb{K})$, that $\operatorname{Str}' H_2(\mathbb{K})$ is the Lie algebra of the Lorentz group of $H_2(\mathbb{K})$, regarded as a Minkowski space–time with $\mathbb{R}I$ as a timelike subspace and the space V as the orthogonal spacelike subspace. The metric g of $H_2(\mathbb{K})$ is then given by

$$g(X, X) = \det X = \alpha\beta - |x|^2 \qquad \text{where } X = \begin{pmatrix} \alpha & x \\ \bar{x} & \beta \end{pmatrix}. \tag{3.25}$$

Thus we have an isomorphism

$$\mathbf{sl}(2, \mathbb{K}_\nu) = \mathbf{so}(\nu+1, 1) \tag{3.26}$$

which incorporates the second column of table 1 and the identification $\mathbf{sl}(2, \mathbb{O}) = \mathbf{so}(9, 1)$. This isomorphism can also be understood by regarding $\mathbf{sl}(2, \mathbb{K})$ as the Lie algebra of the group of projective transformations of the projective line $\mathbb{K}P^1$, defined as the set of equivalence classes of $\mathbb{K}^2$ under the equivalence relation

$$x \sim y \qquad \Leftrightarrow \qquad x_1 x_2^{-1} = y_1 y_2^{-1} \quad \text{or} \quad x_2 = y_2 = 0. \tag{3.27}$$

Thus $\mathbb{K}P^1$ is homeomorphic to $\mathbb{K}$ compactified by the addition of a point at infinity. If we normalise $\boldsymbol{x}$ and $\boldsymbol{y}$ by the condition

$$\boldsymbol{x}^\dagger \boldsymbol{x} = \boldsymbol{y}^\dagger \boldsymbol{y} = 1 \tag{3.28}$$

then (3.27) becomes

$$\boldsymbol{x} \sim \boldsymbol{y} \Leftrightarrow \boldsymbol{x}\boldsymbol{x}^\dagger = \boldsymbol{y}\boldsymbol{y}^\dagger. \tag{3.29}$$

Thus points of $\mathbb{K}P^1$ correspond to certain 2×2 hermitian matrices $X = \boldsymbol{x}\boldsymbol{x}^\dagger$—specifically, the idempotent matrices of rank 1. (The map ϕ taking $\boldsymbol{x}$ satisfying (3.28), i.e. lying on the sphere S^ν, to its equivalence class in $\mathbb{K}P^1$ is the Hopf map ϕ: $S^{2\nu-1} \to S^\nu$, which has fibre $S^{\nu-1}$.) Now an infinitesimal transformation $\delta\boldsymbol{x} = M\boldsymbol{x}$, where M is a 2×2 matrix, corresponds in the representation (3.29) to an infinitesimal transformation of $\mathbb{K}P^1$ given by $\delta X = MX + XM^\dagger$. Comparing with (3.24) suggests that the Lie algebra of infinitesimal projective transformations should be $\mathbf{sl}(2, \mathbb{K})$, and this is confirmed by a more careful treatment in which projective transformations of $\mathbb{K}P^1$ are defined by embedding $\mathbb{K}P^1$ in a projective plane (see Springer (1960) for the octonionic case). On the other hand, a finite projective transformation $\boldsymbol{x} \mapsto U\boldsymbol{x}$ corresponds, in the representation (3.27), to a Möbius transformation

$$x \mapsto (ax+b)(cx+d)^{-1} \tag{3.30}$$

of $\mathbb{K} \cup \{\infty\}$. These are conformal transformations of $\mathbb{K}$, and the conformal group of $\mathbb{K} = \mathbb{R}^\nu$ is isomorphic (Bander and Itzykson 1966) to the Lorentz group $\mathrm{O}(\nu+1, 1)$.

4. 4×4 matrices: de Sitter groups ($s = \nu+2$, $t = 2$)

In § 3 we considered two Lie algebras $\operatorname{Der} \mathbb{J}$ and $\operatorname{Str} \mathbb{J}$ associated with a Jordan algebra $\mathbb{J}$ in the case $\mathbb{J} = H_2(\mathbb{K})$. A third Lie algebra associated with a Jordan algebra was constructed by Kantor (1973) and Koecher (1967); we will call it $\operatorname{Con} \mathbb{J}$ since it is the Lie algebra of a group of conformal transformations of $\mathbb{J}$.

For Koecher's construction the Jordan algebra $\mathbb{J}$ must have an identity. Its structure algebra is then given by (3.18), and has an involutive automorphism $T \to T^*$ which is the identity on Der $\mathbb{J}$ and multiplies every element of $R(\mathbb{J})$ by -1. Now Con $\mathbb{J}$ is defined by

$$\text{Con } \mathbb{J} = \text{Str } \mathbb{J} \oplus \mathbb{J}^2$$

with Str $\mathbb{J}$ a subalgebra and the other Lie brackets given by

$$[T, (x, y)] = (Tx, T^*y), \tag{4.1}$$

$$[(x, 0), (y, 0)] = 0 = [(0, x), (0, y)], \tag{4.2}$$

$$[(x, 0), (0, y)] = 2L_{xy} + 2[L_x, L_y] \in \text{Str } \mathbb{J}, \tag{4.3}$$

for $T \in \text{Str } \mathbb{J}$ and $x, y \in \mathbb{J}$.

For $\mathbb{J} = H_2(\mathbb{K})$ we have, using (3.16) and (2.13),

$$\text{Con } H_2(\mathbb{K}) = [H_2(\mathbb{K})]^2 \oplus L_2'(\mathbb{K}) \oplus \mathbb{R} \oplus \mathbf{so}(\mathbb{K}'). \tag{4.4}$$

An element of $[H_2(\mathbb{K})]^2 \oplus L_2'(\mathbb{K}) \oplus \mathbb{R}$ can be represented as a 4×4 matrix

$$P = \begin{pmatrix} M + \lambda I & X \\ Y & -M^\dagger - \lambda \end{pmatrix} \tag{4.5}$$

with $M \in L_2'(\mathbb{K})$, $X, Y \in H_2(\mathbb{K})$ and $\lambda \in \mathbb{R}$. This matrix satisfies

$$PJ + JP^\dagger = 0, \qquad \text{Tr } P = 0, \tag{4.6}$$

where $J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$. Conversely, any matrix satisfying these conditions can be put in the form (4.5). We will denote the space of such matrices by $\text{Sp}_4'(\mathbb{K})$, so that

$$\text{Con } H_2(\mathbb{K}) \cong \text{Sp}_4'(\mathbb{K}) \oplus \mathbf{so}(\mathbb{K}'). \tag{4.7}$$

In the description of Str $H_2(\mathbb{K})$ as $L_2'(\mathbb{K}) \oplus \mathbf{so}(\mathbb{K}') \oplus \mathbb{R}$ the involution $T \to T^*$ is the map $M \to -M$ on $L_2'(\mathbb{K})$, the identity on $\mathbf{so}(\mathbb{K})$, and -1 on $\mathbb{R}$. It follows from this and the Lie brackets (4.1)–(4.3) that the structure of Con $H_2(\mathbb{K})$ is given by the statements that $\mathbf{so}(\mathbb{K}')$ is a subalgebra which acts on $\text{Sp}_4'(\mathbb{K})$ elementwise, and that the Lie bracket of two matrices $P, Q \in \text{Sp}_4'(\mathbb{K})$ is

$$[P, Q] = (PQ - QP - aI) \oplus (C_a + \tfrac{1}{2}E(P, Q)) \tag{4.8}$$

where $a = \frac{1}{4}\text{Tr}(PQ - QP)$ and $E(P, Q) \in \mathbf{so}(\mathbb{K}')$ is still given by (3.20). We will denote this Lie algebra by $\mathbf{sp}(4, \mathbb{K})$.

On the other hand, Con $H_2(\mathbb{K})$ can be identified as a pseudo-orthogonal, or conformal, Lie algebra. We know that $H_2(\mathbb{K}_\nu)$ carries a Lorentz metric of signature $(\nu + 1, 1)$, and that Str $H_2(\mathbb{K}_\nu)$ is the Lie algebra of its Lorentz group, together with dilations. The automorphism $A \to A^*$ corresponds to time reversal in this Lorentz group, i.e. $A^* = TAT^{-1}$ where T acts on $H_2(\mathbb{K})$ by

$$T\begin{pmatrix} \alpha & x \\ \bar{x} & \beta \end{pmatrix} = \begin{pmatrix} -\beta & x \\ \bar{x} & -\alpha \end{pmatrix}. \tag{4.9}$$

Let $\tau(X) = (X, 0)$ and $\iota(X) = (0, T(X)) \in \text{Con } H_2(\mathbb{K})$; then the Lie brackets (4.1)–(4.3) can be written as

$$\begin{aligned} &[S, \tau(X)] = \tau(S(X)), \qquad [S, \iota(X)] = \iota(S(X)), \\ &[\tau(X), \iota(Y)] = -g(X, Y)I - R(X, Y), \end{aligned} \tag{4.10}$$

where $X, Y \in H_2(\mathbb{K})$, $S \in \mathrm{Str}\, H_2(\mathbb{K})$, g denotes the Lorentz inner product in $H_2(\mathbb{K})$, and $R(X, Y) \in \mathrm{Str}\, H_2(\mathbb{K})$ is defined by

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y. \tag{4.11}$$

The Lie brackets (4.10) define the Lie algebra of the conformal group of $H_2(\mathbb{K}_\nu)$, $\tau(X)$ being a generator of translations, $\iota(X)$ a generator of conformal transformations and I the generator of dilations. Since the conformal group in a space–time of dimension $s+t$ is isomorphic to the Lorentz group $\mathrm{SO}(s+1, t+1)$, we now have an isomorphism

$$\mathbf{sp}(4, \mathbb{K}) \cong \mathbf{so}(\nu+2, 2). \tag{4.12}$$

In the case $\mathbb{K} = \mathbb{C}$ the algebra $\mathbf{sp}(4, \mathbb{C})$ is isomorphic to the Lie algebra of the pseudo-unitary group $\mathrm{SU}(2, 2)$, since there is a unitary equivalence between the antisymmetric matrix J and the matrix $\mathrm{i}G$, where $G = \left(\begin{smallmatrix} I & 0 \\ 0 & -I \end{smallmatrix}\right)$. Thus in this case the isomorphism (4.10) can be written as

$$\mathbf{so}(4, 2) \cong \mathbf{su}(2, 2). \tag{4.13}$$

The results of this and § 3 are summarised in table 3.

5. 3×3 matrices: exceptional groups

Although it is not strictly relevant to our subject of pseudo-orthogonal groups and spinors, it is interesting to see how the constructions of §§ 3 and 4 can be applied to 3×3 matrices. The result is a table of Lie algebras which includes all the exceptional algebras in the Cartan–Killing classification.

The Lie algebras we consider are the derivation, structure and conformal Lie algebras of the Jordan algebras $H_3(\mathbb{K})$. The derivation algebra is determined by the matrix identity (3.5) which shows that the map $X \mapsto [A, X]$ is a derivation of $H_3(\mathbb{K})$ if A is antihermitian and traceless. Together with derivations of $\mathbb{K}$ acting elementwise on the matrices, these are all the derivations of $H_3(\mathbb{K})$, so

$$\mathrm{Der}\, H_3(\mathbb{K}) = \mathrm{ad}\, A_3'(\mathbb{K}) \oplus \mathrm{Der}\, \mathbb{K}. \tag{5.1}$$

This is isomorphic to $\mathbf{sa}(3, \mathbb{K})$ if $\mathbb{K}$ is associative, and in general we use it to define $\mathbf{sa}(3, \mathbb{K})$:

$$\mathbf{sa}(3, \mathbb{K}) = A_3'(\mathbb{K}) \oplus \mathrm{Der}\, \mathbb{K}. \tag{5.2}$$

The Lie brackets are determined by (3.8), which continues to hold for 3×3 matrices; this gives

$$[A, B] = (AB - BA - aI) \oplus (C_a + E(A, B))$$

where $a = \frac{1}{3}\mathrm{Tr}(AB - BA)$. This can be written in terms of the derivations $D(x, y)$ defined by (2.10):

$$[A, B] = (AB - BA - aI) \oplus D(A, B) \tag{5.3}$$

where

$$D(A, B) = \sum_{ij} D(a_{ij}, b_{ji}). \tag{5.4}$$

When $\mathbb{K}$ is the octonions, this algebra $\mathbf{sa}(3, \mathbb{O})$ is the compact form of the exceptional Lie algebra F_4.

Table 3. Isomorphisms of Lie algebras obtained from division algebras $\mathbb{K}$ and Jordan algebras $H_2(\mathbb{K})$. $\mathbf{sq}(n)$ is the compact Lie algebra often denoted by $\mathbf{sp}(n)$.

	$\mathbb{K}=\mathbb{R}$	$\mathbb{K}=\mathbb{C}$	$\mathbb{K}=\mathbb{H}$	$\mathbb{K}=\mathbb{O}$
$\mathbf{so}(\mathbb{K}')=\mathrm{Der}\,\mathbb{K}+C(\mathbb{K}')$		$\mathbf{so}(1)=0$	$\mathbf{so}(3)\cong\mathbf{sq}(1)$	$\mathbf{so}(7)=\mathrm{G}_2\oplus\mathbb{O}'$
$\mathbf{so}(\mathbb{K})=\mathrm{Der}\,\mathbb{K}+L(\mathbb{K}')+R(\mathbb{K}')$		$\mathbf{so}(2)=i\mathbb{R}$	$\mathbf{so}(4)\cong\mathbf{sq}(1)\oplus\mathbf{sq}(1)$	$\mathbf{so}(8)=\mathrm{G}_2\oplus\mathbb{O}'\oplus\mathbb{O}'$
$\mathrm{Der}\,H_2(\mathbb{K})\cong\mathbf{sa}(2,\mathbb{K})$ $\cong\mathbf{so}(\nu+1)$	$\mathbf{so}(2)\cong\mathbf{so}(2)$	$\mathbf{so}(3)\cong\mathbf{su}(2)$	$\mathbf{so}(5)\cong\mathbf{sq}(2)$	$\mathbf{so}(9)$
$\mathrm{Str}\,H_2(\mathbb{K})\cong\mathbf{sl}(2,\mathbb{K})$ $\cong\mathbf{so}(\nu+1,1)$	$\mathbf{so}(2,1)\cong\mathbf{sl}(2,\mathbb{R})$	$\mathbf{so}(3,1)\cong\mathbf{sl}(2,\mathbb{C})$	$\mathbf{so}(5,1)\cong\mathbf{sl}(2,\mathbb{H})$	$\mathbf{so}(9,1)\cong\mathbf{sl}(2,\mathbb{O})$
$\mathrm{Con}\,H_2(\mathbb{K})\cong\mathbf{sp}(4,\mathbb{K})$ $\cong\mathbf{so}(\nu+2,2)$	$\mathbf{so}(3,2)\cong\mathbf{sp}(4,\mathbb{R})$	$\mathbf{so}(4,2)\cong\mathbf{su}(2,2)$	$\mathbf{so}(6,2)\cong\mathbf{sp}(4,\mathbb{H})$	$\mathbf{so}(10,2)\cong\mathbf{sp}(4,\mathbb{O})$

Table 4. The non-compact magic square $L_3(\mathbb{K}_1,\mathbb{K}_2)$. $\tilde{\mathbb{K}}$ denotes the split form of the composition algebra $\mathbb{K}$. Exceptional Lie algebras are specified by their complex type and (in brackets) that of their maximal compact subalgebras.

Comment	$\mathbb{K}_2$ \ $\mathbb{K}_1$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathrm{Der}\,H_3(\mathbb{K})=L_3(\mathbb{K},\mathbb{R})$	$\mathbb{R}$	$\mathbf{so}(3)$	$\mathbf{su}(3)$	$\mathbf{sq}(3)$	F_4
$\mathrm{Str}'\,H_3(\mathbb{K})=L_3(\mathbb{K},\mathbb{C})$	$\tilde{\mathbb{C}}$	$\mathbf{sl}(3,\mathbb{R})$	$\mathbf{sl}(3,\mathbb{C})$	$\mathbf{sl}(3,\mathbb{H})$	$\mathrm{E}_6(\mathrm{F}_4)$
$\mathrm{Con}\,H_3(\mathbb{K})=L_3(\mathbb{K},\mathbb{H})$	$\tilde{\mathbb{H}}$	$\mathbf{sp}(6,\mathbb{R})$	$\mathbf{su}(3,3)$	$\mathbf{sp}(6,\mathbb{H})$	$\mathrm{E}_7(\mathrm{E}_6\oplus\mathbf{so}(2))$
	$\tilde{\mathbb{O}}$	$\mathrm{F}_4(\mathbf{sq}(3)\oplus\mathbf{so}(3))$	$\mathrm{E}_6(\mathbf{su}(6)\oplus\mathbf{so}(3))$	$\mathrm{E}_7(\mathbf{so}(12)\oplus\mathbf{so}(3))$	$\mathrm{E}_8(\mathrm{E}_7\oplus\mathbf{so}(3))$

The structure algebra of $H_3(\mathbb{K})$ is obtained from (3.18), as for $H_2(\mathbb{K})$; factoring out the one-dimensional centre, we obtain

$$\mathrm{Str}'\, H_3(\mathbb{K}) = L_3'(\mathbb{K}) \oplus \mathrm{Der}\, \mathbb{K} = \mathbf{sl}(3, \mathbb{K}) \tag{5.5}$$

(this being a definition of $\mathbf{sl}(3, \mathbb{K})$ for non-associative $\mathbb{K}$). Equation (3.22) continues to hold for 3×3 matrices; hence the Lie brackets in $\mathbf{sl}(3, \mathbb{K})$ are also given by (5.3) and (5.4). When $\mathbb{K} = \mathbb{O}$, this Lie algebra is a non-compact form of the exceptional algebra E_6, the maximal compact subalgebra being F_4.

$\mathbf{sl}(3, \mathbb{K})$ is the Lie algebra of the projective group of the projective plane $\mathbb{K}P^2$, which is embedded in $H_3(\mathbb{K})$ as the set of idempotents with trace 1; $\mathbf{sl}(3, \mathbb{K})$ acts on this plane as in (3.24). When $\mathbb{K} = \mathbb{O}$, this is a non-Desarguian projective plane (Springer 1960).

The Kantor–Koecher construction leads to a Lie algebra

$$\mathrm{Con}\, H_3(\mathbb{K}) = \mathrm{Sp}_6'(\mathbb{K}) \oplus \mathrm{Der}\, \mathbb{K} = \mathbf{sp}(6, \mathbb{K}) \tag{5.6}$$

with Lie brackets given by

$$[P, Q] = (PQ - QP - aI) \oplus \tfrac{1}{2} D(P, Q) \tag{5.7}$$

where $a = \frac{1}{6}\mathrm{Tr}(PQ - QP)$ and $D(P, Q)$ is still given by (5.4) (referring now to 6×6 matrices). When $\mathbb{K} = \mathbb{C}$ this Lie algebra is isomorphic to $\mathbf{su}(3, 3)$; when $\mathbb{K} = \mathbb{O}$ it is a non-compact form of E_7, the maximal compact subalgebra being $\mathrm{E}_6 \oplus \mathbf{so}(2)$.

The algebras constructed in this section are shown in table 4. They can be obtained by a unified construction due to Tits (1966; see Schafer 1966) based on the vector space

$$L_3(\mathbb{K}_1, \mathbb{K}_2) = \mathrm{Der}\, H_3(\mathbb{K}_1) \oplus \mathrm{Der}\, \mathbb{K}_2 \oplus H_3'(\mathbb{K}_1) \otimes \mathbb{K}_2' \tag{5.8}$$

in which $\mathbb{K}_1$ and $\mathbb{K}_2$ are composition algebras. Taking $\mathbb{K}_1$ and $\mathbb{K}_2$ to be division algebras gives the 'magic square' (Freudenthal 1963) of compact Lie algebras. Our non-compact forms can be obtained by taking $\mathbb{K}_1$ to be a division algebra and $\mathbb{K}_2$ to be $\mathbb{R}$ or 'split' (pseudo-orthogonal) forms of $\mathbb{C}$ and $\mathbb{H}$. A fourth row can be obtained by taking $\mathbb{K}_2$ to be the split octonions; this is also shown in table 4.

A construction like Tits's can also be applied to the Jordan algebras $H_2(\mathbb{K})$, yielding a magic square of orthogonal algebras. The Lie algebra in this construction has a form similar to (5.8), namely

$$L_2(\mathbb{K}_1, \mathbb{K}_2) = \mathrm{Der}\, H_2(\mathbb{K}_1) \oplus \mathbf{so}(\mathbb{K}_2') \oplus H_2'(\mathbb{K}_1) \otimes \mathbb{K}_2'. \tag{5.9}$$

The results of § 3 then give

$$L_2(\mathbb{K}_1, \mathbb{K}_2) = \mathbf{so}(\mathbb{K}_1 \oplus \mathbb{K}_2). \tag{5.10}$$

The pseudo-orthogonal algebras constructed in § 3 belong to a non-compact version of this square, giving another row to add to table 3. The full square is shown in table 5.

6. Spinors

In §§ 3 and 4 we described various pseudo-orthogonal groups $\mathrm{SO}(s, t)$ in terms of matrices with entries in a division algebra. These matrices did not necessarily form a representation of the group. In this section we will see how the associated column vectors can be regarded as spinors, carrying a representation of the group and (in

Table 5. The non-compact magic square $L_2(\mathbb{K}_1, \mathbb{K}_2)$.

$\mathbb{K}_2$ \ $\mathbb{K}_1$	$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
$\mathbb{R}$	$\mathbf{so}(2)$	$\mathbf{so}(3)$	$\mathbf{so}(5)$	$\mathbf{so}(9)$
$\tilde{\mathbb{C}}$	$\mathbf{so}(2,1)$	$\mathbf{so}(3,1)$	$\mathbf{so}(5,1)$	$\mathbf{so}(9,1)$
$\tilde{\mathbb{H}}$	$\mathbf{so}(3,2)$	$\mathbf{so}(4,2)$	$\mathbf{so}(6,2)$	$\mathbf{so}(10,2)$
$\tilde{\mathbb{O}}$	$\mathbf{so}(5,4)$	$\mathbf{so}(6,4)$	$\mathbf{so}(8,4)$	$\mathbf{so}(12,4)$

some cases) a representation of the Clifford algebra. Since the division algebras are real algebras, these are Majorana spinors. We will construct the Dirac γ-matrices in each case, using the following notation: if V is the pseudo-orthogonal space being considered, with metric g, and S is the spinor space, then for each $v \in V$ we have a linear map $\gamma(v): S \to S$. These $\gamma(v)$ are the Dirac matrices; they satisfy

$$\{\gamma(v), \gamma(w)\} = 2g(v, w). \tag{6.1}$$

For the Euclidean spaces it is sometimes necessary to take g to be negative definite to get real Dirac matrices. In half of the cases we will note that there is 'no real representation of the Clifford algebra'. In such cases it would be possible to construct a complex representation of the Clifford algebra for $\mathbb{K} = \mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$, but this does not fit naturally into our approach; and there is no representation at all for $\mathbb{K} = \mathbb{O}$.

As before, ν denotes the dimension of the division algebra $\mathbb{K}$, with possible values 1, 2, 4 and 8.

SO($\nu - 1$)

This is the orthogonal group of the Euclidean space $V = \mathbb{K}'$. For spinor space we take $S = \mathbb{K}$; the Clifford algebra is represented by the operations of multiplication in $\mathbb{K}$, i.e.

$$\gamma(a) = L_a \qquad (a \in \mathbb{K}'). \tag{6.2}$$

Equation (6.1) becomes

$$L_a L_b + L_b L_a = -2\langle a, b\rangle \tag{6.3}$$

which follows from the definition (2.5) of the inner product and the alternative law (2.7).

From this representation of the Clifford algebra we obtain a spin representation of the group. As usual, it is easiest to describe the representation of the Lie algebra: the $\mathbf{so}(\mathbb{K}')$ element $Q(a, b)$ defined by

$$Q(a, b)c = \langle b, c\rangle a - \langle a, c\rangle b \qquad (a, b, c \in \mathbb{K}') \tag{6.4}$$

(the generator of rotations in the plane of a and b) is represented by

$$Q(a, b)^{\#} = -\tfrac{1}{2}[L_a, L_b]. \tag{6.5}$$

The only interesting cases are $\nu = 4$ and 8, $\mathbb{K} = \mathbb{H}$ and $\mathbb{O}$. In the first case we have a real four-dimensional representation of $\mathbf{so}(3)$; by taking one of the quaternion units as the imaginary unit we can identify $\mathbb{H} = \mathbb{C}^2$ and this representation becomes the usual complex two-dimensional spin representation. In the second case we have a real eight-dimensional representation of $\mathbf{so}(7)$, which is the spin representation.

SO(ν)

In this case the Euclidean space V is $\mathbb{K}$. $\mathbb{K}$ also carries spinor representations of $\mathbf{so}(\nu)$, but there is no real representation of the Clifford algebra.

For each $x, y \in \mathbb{K}$, let $Q(a, y) \in \mathbf{so}(\mathbb{K})$ be the generator of rotations in the plane of x and y, as in (6.4). Define $Q(x, y)^\sharp$ and $Q(x, y)^\flat \in \mathbf{so}(\mathbb{K})$ by

$$Q(x, y)^\sharp z = \tfrac{1}{4}[x(\bar{y}z) - y(\bar{x}z)], \tag{6.6}$$

$$Q(x, y)^\flat z = \tfrac{1}{4}[(z\bar{y})x - (z\bar{x})y]. \tag{6.7}$$

Then $T \mapsto T^\sharp$ and $T \mapsto T^\flat$ are representations of $\mathbf{so}(\mathbb{K})$ acting on $\mathbb{K}$ which are not equivalent to the self-representation. They satisfy the triality equations

$$T(xy) = (T^\sharp x)y + x(\overline{T^\flat} y), \tag{6.8}$$

$$T^\sharp(xy) = (Tx)y + x(T^\flat y), \tag{6.9}$$

$$T^\flat(xy) = (\overline{T^\sharp} x)y + x(Ty), \tag{6.10}$$

where the conjugate $\bar{F}$ of any linear map $F: \mathbb{K} \mapsto \mathbb{K}$ is defined by

$$\bar{F}(x) = \overline{F(\bar{x})}. \tag{6.11}$$

Note that for $T \in \mathbf{so}(\mathbb{K}')$ we have

$$\bar{T} = T \qquad \text{and} \qquad \overline{T^\sharp} = T^\flat. \tag{6.12}$$

If $\mathbb{K}$ is associative $\mathbf{so}(\mathbb{K})$ is spanned by the maps of left and right multiplication by elements of $\mathbb{K}'$, and the representations $\sharp$ and $\flat$ are given by

$$\begin{aligned} L_a^\sharp &= \tfrac{1}{4}\nu L_a, & R_a^\sharp &= (1 - \tfrac{1}{4}\nu)L_a \\ L_a^\flat &= (1 - \tfrac{1}{4}\nu)R_a, & R_a^\flat &= \tfrac{1}{4}\nu R_a \end{aligned} \qquad (a \in \mathbb{K}'). \tag{6.13}$$

The case $\mathbb{K} = \mathbb{R}$ is trivial. For $\mathbb{K} = \mathbb{C}$ the representations $T^\sharp$ and $T^\flat$ coincide: if T is multiplication by i, $T^\sharp$ and $T^\flat$ are multiplication by $\tfrac{1}{2}$i. The corresponding double-valued representation of SO(2) represents each rotation by a rotation through half the angle.

For $\mathbb{K} = \mathbb{H}$, $T^\sharp$ and $T^\flat$ are projections onto the two commuting subalgebras in the decomposition (2.18). The corresponding double-valued representations of SO(4) are given by:

if

$$R(x) = uzv \qquad (R \in \mathrm{SO}(4), x \in \mathbb{H}, u, v \in S^3)$$

then

$$R^\sharp z = \pm uz, \qquad R^\flat z = \pm zv. \tag{6.14}$$

These are well defined as representations of Spin(4).

In the non-associative case $\mathbb{K} = \mathbb{O}$ the Lie algebra $\mathbf{so}(\mathbb{K})$ is spanned by derivations as well as left and right multiplication maps, and we have

$$D^\sharp = D^\flat = D \qquad (D \in \mathrm{Der}\, \mathbb{O}), \tag{6.15}$$

$$\begin{aligned} L_a^\sharp &= L_a + R_a, & R_a^\sharp &= -R_a \\ L_a^\sharp &= -L_a, & R_a^\flat &= L_a + R_a \end{aligned} \qquad (a \in \mathbb{O}'). \tag{6.16}$$

For future reference we note that it follows (using (2.11)) that

$$E(a,b)^\sharp = E(a,b) - R_{[a,b]}, \qquad E(a,b)^\flat = E(a,b) + L_{[a,b]}. \tag{6.17}$$

Equations (6.15) and (6.16) define the triality representations (Ramond 1976, Schafer 1966) of SO(8). The corresponding representations of SO(8) are again double valued. Further details of triality are given by van der Blij and Springer (1960) and Porteous (1982).

SO(ν+1)

In § 3 the Lie algebra $\mathbf{so}(\nu+1)$ was realised by means of 2×2 traceless antihermitian matrices over $\mathbb{K}$ together with antisymmetric maps of $\mathbb{K}'$:

$$\mathbf{so}(\nu+1) = A_2'(\mathbb{K}) \oplus \mathbf{so}(\mathbb{K}') \tag{6.18}$$

(see (3.7) and (3.15)). We take S to be the space of 2×1 column vectors with entries in $\mathbb{K}$, and define an action ρ of $A_2'(\mathbb{K})\oplus\mathbf{so}(\mathbb{K}')$ on S by

$$A \in A_2'(\mathbb{K}) \Rightarrow \rho(A)\boldsymbol{x} = A\boldsymbol{x} \qquad \text{(matrix multiplication)}, \tag{6.19}$$

$$T \in \mathrm{so}(\mathbb{K}') \Rightarrow \rho(T)\boldsymbol{x} = T^\sharp \boldsymbol{x} \qquad \text{(componentwise action)}. \tag{6.20}$$

Then from (6.9) and (6.12) we have

$$[\rho(T), \rho(A)] = \rho(TA). \tag{6.21}$$

Moreover there is a matrix identity

$$A(B\boldsymbol{x}) - B(A\boldsymbol{x}) = (AB - BA)\boldsymbol{x} + E(A,B)\boldsymbol{x} \tag{6.22}$$

where $E(A,B)\in\mathbf{so}(\mathbb{K}')$ is defined in (3.9), which is valid for any traceless 2×2 matrices A, B with entries in an alternative algebra, and from which it follows that

$$[\rho(A), \rho(B)] = (AB - BA)\boldsymbol{x} - a\boldsymbol{x} + (C_a + E(A,B)^\sharp)\boldsymbol{x} \tag{6.23}$$

where $a = \frac{1}{2}\mathrm{Tr}(AB - BA)$. (The proof is different for different $\mathbb{K}$; for $\mathbb{R}$ and $\mathbb{C}$ we have $C_a = E(A,B) = 0$, for $\mathbb{H}$ we have $E(A,B) = 0$ and $C_a^\sharp = L_a$, and for $\mathbb{O}$ we have $C_a^\sharp = L_a + 2R_a$ and $E(A,B)^\sharp = E(A,B) - 2R_a$ from (6.16)–(6.17).) Thus

$$[\rho(A), \rho(B)] = \rho([A,B]) \tag{6.24}$$

where the bracket on the right-hand side is the $\mathbf{sa}(2,\mathbb{K})$ bracket defined by (3.11); so ρ is a representation of the Lie algebra $\mathbf{sa}(2,\mathbb{K}) \cong \mathbf{so}(\nu+1)$.

S also carries a real representation of the Clifford algebra for the positive definite metric, given by the traceless hermitian matrices which defined the orthogonal space in the first place. This is a representation by virtue of the identity

$$X(Y\boldsymbol{x}) + Y(X\boldsymbol{x}) = (XY + YX)\boldsymbol{x} \tag{6.25}$$

which holds for hermitian X and Y over any composition algebra.

For $\mathbb{K}=\mathbb{R}$ the representation ρ gives the half-angle representation of SO(2); for $\mathbb{K}=\mathbb{C}$ it gives the spin-$\frac{1}{2}$ representation of SO(3). For $\mathbb{K}=\mathbb{H}$ we obtain an eight-dimensional real representation of SO(5) which, by means of the identification $\mathbb{H}=\mathbb{C}^2$, can be identified with the four-dimensional complex (spin) representation. Finally, $\mathbb{K}=\mathbb{O}$ gives the 16-dimensional spin representation of SO(9).

SO($\nu+1, 1$)

Since the identity (6.22) holds for all traceless matrices, the considerations on $\mathbf{sa}(2, \mathbb{K}) \cong \mathbf{so}(\nu+1)$ can be extended to $\mathbf{sl}(2, \mathbb{K}) \cong \mathbf{so}(\nu+1, 1)$ to give a double-valued representation on the space $\mathbb{K}^2$. For $\mathbb{K}=\mathbb{R}$, $\mathbb{C}$ and $\mathbb{H}$ these are the familiar representations implied by the isomorphisms in the fourth row of table 2; for $\mathbb{K}=\mathbb{O}$ we obtain an identification of SO(9, 1) spinors as pairs of octonions. There is no real representation of the Clifford algebra.

SO($\nu+2, 2$)

In § 4 the Lie algebra $\mathbf{so}(\nu+2, 2)$ was realised by means of certain 4×4 matrices over $\mathbb{K}$, together with antisymmetric maps of $\mathbb{K}'$:

$$\mathbf{so}(\nu+2, 2) = \mathbf{sp}(4, \mathbb{K}) = \mathrm{Sp}_4'(\mathbb{K}) \oplus \mathbf{so}(\mathbb{K}') \tag{6.26}$$

(see (4.7) and (4.12)). We take S to be the space of 4×1 column vectors with entries in $\mathbb{K}$, and define an action σ of $\mathbf{sp}(4, \mathbb{K})$ by letting $\mathrm{Sp}_4'(\mathbb{K})$ act by matrix multiplication and $\mathbf{so}(\mathbb{K}')$ componentwise in the $\sharp$ representation, as in (6.19) and (6.20). As for $\mathbf{so}(\nu+1)$, this gives the correct commutators between $\mathrm{Sp}_4'(\mathbb{K})$ and $\mathbf{so}(\mathbb{K}')$ for a representation of $\mathbf{sp}(4, \mathbb{K})$. Also, from (6.22) and (6.25) and their consequences

$$A(X\boldsymbol{x}) + X(A^{\dagger}\boldsymbol{x}) = (AX + XA^{\dagger})\boldsymbol{x}, \tag{6.27}$$

$$X(Y\boldsymbol{x}) = (XY)\boldsymbol{x} + \tfrac{1}{2}E(X, Y)\boldsymbol{x}, \tag{6.28}$$

which hold for 2×2 matrices if A is traceless and X and Y are hermitian, we find that

$$M(N\boldsymbol{v}) - N(M\boldsymbol{v}) = (MN - NM)\boldsymbol{v} + \tfrac{1}{2}E(M, N)\boldsymbol{v} \tag{6.29}$$

for $M, N \in \mathrm{Sp}_4'(\mathbb{K})$ and $\boldsymbol{v} \in \mathbb{K}^4$. It follows as for $\mathbf{so}(\nu+1)$ that σ is a representation of $\mathbf{sp}(4, \mathbb{K})$.

There is no representation of the Clifford algebra in general, but this space does carry a representation of the Clifford algebra of $\mathbf{so}(\nu+1, 1)$. The Minkowski space $\mathbb{R}^{\nu+2}$ is realised as $H_2(\mathbb{K})$, with metric g determined by

$$g(X, X) = \det X; \tag{6.30}$$

the Dirac matrices are then given as 4×4 matrices by

$$\begin{aligned} \gamma(X) &= \begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix} \qquad \text{for } X \in H_2'(\mathbb{K}), \\ \gamma(1) &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{6.31}$$

These satisfy

$$\{\gamma(X), \gamma(Y)\} = 2g(X, Y)$$

and if they act on $\mathbb{K}^4$ by matrix multiplication, (6.25) guarantees that they form a representation of the Clifford algebra.

For $\mathbb{K}=\mathbb{C}$ this gives the usual Dirac spinors and (6.31) is a standard representation of the γ-matrices. The representation of SO(4, 2) is the twistor representation of the conformal group of Minkowski space. For $\mathbb{K}=\mathbb{H}$ we obtain an analogous construction of eight-component Dirac spinors for SO(5, 1) (having made the usual identification $\mathbb{H} = \mathbb{C}^2$). $\mathbb{K} = \mathbb{O}$ gives a set of real Dirac matrices for SO(9, 1) and a real 32-dimensional representation of SO(10, 2).

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